VIDYA BHAWAN BALIKA VIDYA PITH

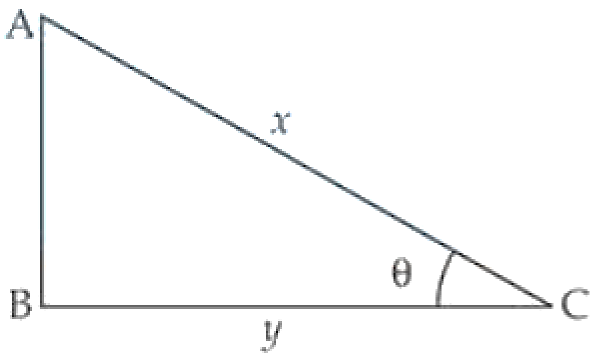
शक्तिउत्थानआश्रमलखीसरायबिहार

Class-12 Sub-.Maths Date 20.06..2021

Long Answer (L.A.)

25. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is π/3.

Solution:



Let ΔABC be the right-angled triangle in which ∠B = 90°

Let AC = x, BC = y

So, AB = $\sqrt{(x^2 + y^2)}$

∠ACB = θ

Let z = x + y (given)

Now, the area of ΔABC = ½ x AB x BC

$$\Rightarrow \quad A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \quad \Rightarrow A = \frac{1}{2} y \cdot \sqrt{(Z-y)^2 - y^2}$$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 \left[(Z-y)^2 - y^2\right] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

So, $P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4} [y^2 Z^2 - 2Zy^3]$ $\quad [A^2 = P]$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4} [2yZ^2 - 6Zy^2] \quad \ldots(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$$\therefore \quad \frac{1}{4}(2yZ^2 - 6Zy^2) = 0$$

$$\frac{2yZ}{4}(Z - 3y) = 0 \quad \Rightarrow \quad yZ(Z - 3y) = 0$$

$$yZ \neq 0 \qquad (\because \quad y \neq 0 \text{ and } Z \neq 0)$$

$$\therefore \; Z - 3y = 0$$

$$y = \frac{Z}{3} \quad \Rightarrow y = \frac{x + y}{3} \qquad (\because \quad Z = x + y)$$

$$3y = x + y \quad \Rightarrow 3y - y = x \quad \Rightarrow 2y = x$$

$$\frac{y}{x} = \frac{1}{2} \quad \Rightarrow \cos\theta = \frac{1}{2}$$

Thus, $\quad \theta = \frac{\pi}{3}$

Differentiating eq. (i) w.r.t. y, we have $\frac{d^2P}{dy^2} = \frac{1}{4}[2Z^2 - 12Zy]$

$$\frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} = \frac{1}{4}\left[2Z^2 - 12Z \cdot \frac{Z}{3}\right]$$

$$= \frac{1}{4}[2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima}$$

$$\Rightarrow \quad A = \frac{1}{2} y \cdot \sqrt{x^2 - y^2} \quad \Rightarrow A = \frac{1}{2} y \cdot \sqrt{(Z-y)^2 - y^2}$$

Squaring both sides, we get

$$A^2 = \frac{1}{4} y^2 \left[(Z-y)^2 - y^2\right] \Rightarrow A^2 = \frac{1}{4} y^2 [Z^2 + y^2 - 2Zy - y^2]$$

So, $P = \frac{1}{4} y^2 [Z^2 - 2Zy] \Rightarrow P = \frac{1}{4}[y^2Z^2 - 2Zy^3] \qquad [A^2 = P]$

Differentiating both sides w.r.t. y we get

$$\frac{dP}{dy} = \frac{1}{4}[2yZ^2 - 6Zy^2] \qquad ...(i)$$

For local maxima and local minima, $\frac{dP}{dy} = 0$

$$\therefore \quad \frac{1}{4}(2yZ^2 - 6Zy^2) = 0$$

$$\frac{2yZ}{4}(Z - 3y) = 0 \quad \Rightarrow \quad yZ(Z - 3y) = 0$$

$$yZ \neq 0 \qquad (\because \quad y \neq 0 \text{ and } Z \neq 0)$$

$$\therefore \quad Z - 3y = 0$$

$$y = \frac{Z}{3} \quad \Rightarrow y = \frac{x+y}{3} \qquad (\because \quad Z = x + y)$$

$$3y = x + y \quad \Rightarrow 3y - y = x \quad \Rightarrow 2y = x$$

$$\frac{y}{x} = \frac{1}{2} \quad \Rightarrow \cos\theta = \frac{1}{2}$$

Thus, $\theta = \frac{\pi}{3}$

Differentiating eq. (i) w.r.t. y, we have $\frac{d^2P}{dy^2} = \frac{1}{4}[2Z^2 - 12Zy]$

$$\frac{d^2P}{dy^2} \text{ at } y = \frac{Z}{3} = \frac{1}{4}\left[2Z^2 - 12Z \cdot \frac{Z}{3}\right]$$

$$= \frac{1}{4}[2Z^2 - 4Z^2] = \frac{-Z^2}{2} < 0 \text{ Maxima}$$

Therefore, the area of the given triangle is maximum when the angle between its hypotenuse and a side is π/3.

26. Find the points of local maxima, local minima and the points of inflection of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also find the corresponding local maximum and local minimum values.

Solution:

Given, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Differentiating the function,

$f'(x) = 5x^4 - 20x^3 + 15x^2$

For local maxima and local minima, $f'(x) = 0$

$$5x^4 - 20x^3 + 15x^2 = 0 \quad \Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow 5x^2(x^2 - 3x - x + 3) = 0 \quad \Rightarrow x^2(x-3)(x-1) = 0$$

$$\therefore x = 0,\ x = 1 \text{ and } x = 3$$

Now $\qquad f''(x) = 20x^3 - 60x^2 + 30x$

$\Rightarrow \qquad f''(x)_{\text{at } x=0} = 20(0)^3 - 60(0)^2 + 30(0) = 0$ which is neither maxima nor minima.

$\therefore \quad f(x)$ has the point of inflection at $x = 0$

$$f''(x)_{\text{at } x=1} = 20(1)^3 - 60(1)^2 + 30(1)$$
$$= 20 - 60 + 30 = -10 < 0 \text{ Maxima}$$
$$f''(x)_{\text{at } x=3} = 20(3)^3 - 60(3)^2 + 30(3)$$
$$= 540 - 540 + 90 = 90 > 0 \text{ Minima}$$

The maximum value of the function at x = 1

$f(x) = (1)^5 - 5(1)^4 + 5(1)^3 - 1$

$= 1 - 5 + 5 - 1 = 0$

The minimum value at x = 3 is

$f(x) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$

$= 243 - 405 + 135 - 1$

$= 378 - 406 = -28$

Therefore, the function has its maxima at $x = 1$ and the maximum value = 0 and its has minimum value at $x = 3$ and its minimum value is -28.